

# DATA DEPENDENCE OF APPROXIMATE CURRENT-VORTEX SHEETS NEAR THE ONSET OF INSTABILITY

ALESSANDRO MORANDO, PAOLO SECCHI, AND PAOLA TREBESCHI

**ABSTRACT.** The paper is concerned with the free boundary problem for 2D current-vortex sheets in ideal incompressible magneto-hydrodynamics near the transition point between the linearized stability and instability. In order to study the dynamics of the discontinuity near the onset of the instability, Hunter and Thoo [4] have introduced an asymptotic quadratically nonlinear integro-differential equation for the amplitude of small perturbations of the planar discontinuity.

The local-in-time existence of smooth solutions to the Cauchy problem for the amplitude equation was shown in [8, 7]. In the present paper we prove the continuous dependence in strong norm of solutions on the initial data. This completes the proof of the well-posedness of the problem in the classical sense of Hadamard.

## 1. INTRODUCTION

In the present paper we consider the following equation

$$\varphi_{tt} - \mu\varphi_{xx} = \left( \frac{1}{2} \mathbb{H}[\phi^2]_{xx} + \phi\varphi_{xx} \right)_x, \quad \phi = \mathbb{H}[\varphi], \quad (1)$$

where the unknown is the scalar function  $\varphi = \varphi(t, x)$ , where  $t$  denotes the time,  $x \in \mathbb{R}$  is the space variable and  $\mathbb{H}$  denotes the Hilbert transform with respect to  $x$ , and  $\mu$  is a constant. Hunter and Thoo [4] have derived this asymptotic equation in order to study the dynamics of 2D current-vortex sheets in ideal incompressible magneto-hydrodynamics, near the transition point between the linearized stability and instability.

Equation (1) is an integro-differential evolution equation in one space dimension, with quadratic nonlinearity. This is a nonlocal equation of order two: in fact, it may also be written as

$$\varphi_{tt} - \mu\varphi_{xx} = ([\mathbb{H}; \phi] \partial_x \phi_x + \mathbb{H}[\phi_x^2])_x, \quad (2)$$

where  $[\mathbb{H}; \phi] \partial_x$  is a pseudo-differential operator of order zero. This alternative form (2) shows that (1) is an equation of second order, due to a cancelation of the third order spatial derivatives appearing in (1).

Equation (1) also admits the alternative spatial form

$$\varphi_{tt} - (\mu - 2\phi_x) \varphi_{xx} + \mathcal{Q}[\varphi] = 0, \quad (3)$$

where

$$\mathcal{Q}[\varphi] := -3 [\mathbb{H}; \phi_x] \phi_{xx} - [\mathbb{H}; \phi] \phi_{xxx}. \quad (4)$$

The alternative form (3) puts in evidence the difference  $\mu - 2\phi_x$  which has a meaningful role. In fact it can be shown that the linearized operator about a given basic state is elliptic and (1) is locally linearly ill-posed in points where

$$\mu - 2\phi_x < 0.$$

On the contrary, in points where

$$\mu - 2\phi_x > 0 \quad (5)$$

---

*Date:* January 15, 2016.

*2010 Mathematics Subject Classification.* 35Q35, 76E17, 76E25, 35R35, 76B03.

*Key words and phrases.* Magneto-hydrodynamics, incompressible fluids, current-vortex sheets, interfacial stability and instability.

The authors are supported by the national research project PRIN 2012 “Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations: theoretical and applicative aspects”.

the linearized operator is hyperbolic and (1) is locally linearly well-posed, see [4]. In this case we can think of (3) as a nonlinear perturbation of the wave equation.

The local-in-time existence of smooth solutions to the Cauchy problem for (1), under the above stability condition (5), was shown in [8, 7]. In the present paper we prove the continuous dependence in strong norm of solutions on the initial data. This completes the proof of the well-posedness of (1) in the classical sense of Hadamard, after existence and uniqueness.

As written in Kato's paper [5], this part may be the most difficult one, when dealing with hyperbolic problems. Our method is somehow inspired by Beirão da Veiga's perturbation theory for the compressible Euler equations [1, 2], and its application to the problem of convergence in strong norm of the incompressible limit, see [3, 9]. Instead of directly estimating the difference between the given solution and the solutions to the problems with approximating initial data, the main idea is to use a triangularization with the more regular solution to a suitably chosen close enough problem.

Let us consider the initial value problem for the equation (1) supplemented by the initial condition

$$\varphi|_{t=0} = \varphi^{(0)}, \quad \partial_t \varphi|_{t=0} = \varphi^{(1)}, \quad (6)$$

for sufficiently smooth initial data  $\varphi^{(0)}, \varphi^{(1)}$  satisfying the stability condition

$$\mu - 2\phi_x^{(0)} > 0, \quad \phi^{(0)} := \mathbb{H}[\varphi^{(0)}].$$

For the sake of convenience, in the paper the unknown  $\varphi = \varphi(t, x)$  is a scalar function of the time  $t \in \mathbb{R}^+$  and the space variable  $x$ , ranging on the one-dimensional torus  $\mathbb{T}$  (that is  $\varphi$  is periodic in  $x$ ). For all notation we refer to the following Section 2.

In [7] we prove the following existence theorem.

**Theorem 1.** *Let  $s \geq 3$  be a real number. Assume  $\varphi^{(0)} \in H^s(\mathbb{T})$ ,  $\varphi^{(1)} \in H^{s-1}(\mathbb{T})$ , and let  $\varphi^{(0)}, \varphi^{(1)}$  have zero spatial mean. Given  $0 < \delta < \mu$ , there exist  $0 < R \leq 1$ , and constants  $C_1 > 0$ ,  $C_2 > 0$  such that, if*

$$\|\varphi_x^{(0)}\|_{H^2(\mathbb{T})}^2 + \|\varphi^{(1)}\|_{H^2(\mathbb{T})}^2 < R^2, \quad (7)$$

*there exists a unique solution  $\varphi \in C(I_0; H^s(\mathbb{T})) \cap C^1(I_0; H^{s-1}(\mathbb{T}))$  of the Cauchy problem (1), (6) defined on the time interval  $I_0 = [0, T_0)$ , where*

$$T_0 = C_1 \left( \|\varphi_x^{(0)}\|_{H^2(\mathbb{T})}^2 + \|\varphi^{(1)}\|_{H^2(\mathbb{T})}^2 \right)^{-1/2}. \quad (8)$$

*The solution  $\varphi$  has zero spatial mean and satisfies, for all  $t \in I_0$ ,*

$$\mu - 2\phi_x \geq \delta, \quad (9)$$

$$\|\varphi(t)\|_{H^s(\mathbb{T})}^2 + \|\varphi_t(t)\|_{H^{s-1}(\mathbb{T})}^2 \leq C_2 \left( \|\varphi^{(0)}\|_{H^s(\mathbb{T})}^2 + \|\varphi^{(1)}\|_{H^{s-1}(\mathbb{T})}^2 \right). \quad (10)$$

Notice that the size of the existence time interval depends on the  $H^3, H^2$  norms of the initial data, for all  $s \geq 3$ . Every solution, with the regularity as in the statement of Theorem 1, has the additional regularity

$$\varphi \in C^2(I_0; H^{s-2}(\mathbb{T})),$$

following from equation (1) and suitable commutator estimates, see [7], Corollary 16.

**1.1. The main result.** Now we state the main result of this paper about the continuous dependence in strong norm of solutions on the initial data.

**Theorem 2.** *Let  $s \geq 3$  be a real number and  $\mu > \delta > 0$ . Let us consider  $\varphi^{(0)} \in H^s(\mathbb{T})$ ,  $\varphi^{(1)} \in H^{s-1}(\mathbb{T})$  and sequences  $\{\varphi_n^{(0)}\}_{n \in \mathbb{N}} \subset H^s(\mathbb{T})$ ,  $\{\varphi_n^{(1)}\}_{n \in \mathbb{N}} \subset H^{s-1}(\mathbb{T})$ , where all functions  $\varphi^{(0)}, \varphi^{(1)}, \varphi_n^{(0)}, \varphi_n^{(1)}$  have zero spatial mean. Assume that*

$$\varphi_n^{(0)} \rightarrow \varphi^{(0)} \quad \text{strongly in } H^s(\mathbb{T}), \quad \varphi_n^{(1)} \rightarrow \varphi^{(1)} \quad \text{strongly in } H^{s-1}(\mathbb{T}), \quad \text{as } n \rightarrow +\infty.$$

*Let  $T > 0$  and set  $I = [0, T]$ . Assume that there exists a unique solution  $\varphi \in C(I; H^s(\mathbb{T})) \cap C^1(I; H^{s-1}(\mathbb{T}))$  of problem (1), (6) with initial data  $\varphi^{(0)}, \varphi^{(1)}$ , and that for all  $n$  there exist unique solutions  $\varphi_n \in C(I; H^s(\mathbb{T})) \cap C^1(I; H^{s-1}(\mathbb{T}))$  of (1), (6) with initial data  $\varphi_n^{(0)}, \varphi_n^{(1)}$ . All solutions satisfy (9), (10) on  $I$  (with corresponding initial data in the r.h.s.) for a given constant  $C_2$  independent of  $n$ .*

Then

$$\varphi_n \rightarrow \varphi \quad \text{strongly in } C(I; H^s(\mathbb{T})) \cap C^1(I; H^{s-1}(\mathbb{T})), \text{ as } n \rightarrow +\infty. \quad (11)$$

Here we are assuming that all solutions  $\varphi, \varphi_n$  are defined on the same time interval  $I = [0, T]$ , where  $T$  is arbitrarily given, neither necessarily small nor given by (8).

Using the a priori estimate (10) and standard arguments, it is rather easy to show the continuous dependence of solutions on the initial data in the topology of  $C(I; H^{s-\varepsilon}(\mathbb{T})) \cap C^1(I; H^{s-1-\varepsilon}(\mathbb{T}))$ , for all small enough  $\varepsilon > 0$ . Instead, in Theorem 2 we prove the continuous dependence precisely in the topology of  $C(I; H^s(\mathbb{T})) \cap C^1(I; H^{s-1}(\mathbb{T}))$ , i.e. in the same function space where we show the existence. From Theorem 1 and Theorem 2 we obtain that the initial value problem (1), (6) is well-posed in  $H^s$  in the classical sense of Hadamard.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and give some preliminary technical results. In Section 3 we prove our main Theorem 2.

## 2. NOTATIONS AND PRELIMINARY RESULTS

**2.1. Notations.** In the paper we denote by  $C$  generic positive constants, that may vary from line to line or even inside the same formula.

Let  $\mathbb{T}$  denote the one-dimensional torus defined as  $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ . As usual, all functions defined on  $\mathbb{T}$  can be considered as  $2\pi$ -periodic functions on  $\mathbb{R}$ .

All functions  $f : \mathbb{T} \rightarrow \mathbb{C}$  can be expanded in terms of Fourier series as

$$f(x) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{ikx},$$

where  $\{\widehat{f}(k)\}_{k \in \mathbb{Z}}$  are the Fourier coefficients defined by

$$\widehat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}. \quad (12)$$

For positive real numbers  $s$ ,  $H^s = H^s(\mathbb{T})$  denotes the Sobolev space of order  $s$  on  $\mathbb{T}$ , defined to be the set of functions  $f : \mathbb{T} \rightarrow \mathbb{C}$  such that

$$\|f\|_{H^s}^2 := \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 < +\infty, \quad (13)$$

where it is set

$$\langle k \rangle := (1 + |k|^2)^{1/2}. \quad (14)$$

The function  $\|\cdot\|_{H^s}$  defines a norm on  $H^s$ , associated to the inner product

$$(f, g)_{H^s} := \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \widehat{f}(k) \overline{\widehat{g}(k)},$$

which turns  $H^s$  into a Hilbert space. For  $s = 0$  one has  $H^0 = L^2$ . The  $L^2$  norm will simply be denoted by  $\|\cdot\|$ . From (13) we have

$$\|f\|_{H^s} = \|\langle \partial_x \rangle^s f\|,$$

where  $\langle \partial_x \rangle^s$  is the Fourier multiplier of symbol  $\langle k \rangle^s$ , defined by

$$\widehat{\langle \partial_x \rangle^s u}(k) = \langle k \rangle^s \widehat{u}(k), \quad \forall k \in \mathbb{Z}.$$

We will work with functions with zero spatial mean, so that  $\widehat{f}(0) = 0$ . For these functions, we easily obtain from (13) the Poincaré inequality

$$\|f\|_{H^s} \leq \sqrt{2} \|f_x\|_{H^{s-1}} \quad \forall s \geq 1. \quad (15)$$

For  $T > 0$  and  $j \in \mathbb{N}$ , we denote by  $C^j([0, T]; \mathcal{X})$  the space of  $j$  times continuously differentiable functions  $f : \mathbb{R} \rightarrow \mathcal{X}$ .

**2.2. Preliminary results.** Let us consider the Cauchy problem

$$\begin{cases} \psi_{tt} - (\mu - 2\phi_x) \psi_{xx} = F, & t \in I, x \in \mathbb{T}, \\ \psi|_{t=0} = \psi^{(0)}, \quad \partial_t \psi|_{t=0} = \psi^{(1)} & x \in \mathbb{T}, \end{cases} \quad (16)$$

with unknown the scalar function  $\psi = \psi(t, x)$ , and where  $\phi = \mathbb{H}[\varphi]$  is the Hilbert transform of a given function  $\varphi$ , sufficiently smooth, with zero mean and such that

$$\mu - 2\phi_x \geq \delta \quad t \in I, x \in \mathbb{T}, \quad (17)$$

for given constants  $\mu > \delta > 0$ . Let us define

$$\mathcal{E}(t) := \left( \|\psi_t(t)\|^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\psi_x(t)|^2 dx \right)^{1/2}.$$

**Lemma 3.** *Let  $\varphi \in C(I; H^3) \cap C^1(I; H^2)$  be given and satisfying (17). For all  $\psi^{(0)} \in H^1$ ,  $\psi^{(1)} \in L^2$ , both functions with zero mean, and  $F \in L^1(I; L^2)$  there exists a unique solution  $\psi \in C(I; H^1) \cap C^1(I; L^2)$ , of (16), with zero mean and such that*

$$\frac{d}{dt} \mathcal{E} \leq C (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3}) \mathcal{E} + \|F\|, \quad (18)$$

for every  $t \in I$ .

*Proof.* To obtain (18) we multiply the equation in (16) by  $\psi_t$  and integrate over  $\mathbb{T}$ . Integrating by parts gives

$$\frac{1}{2} \frac{d}{dt} \left( \|\psi_t\|^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\psi_x|^2 dx \right) = 2 \int_{\mathbb{T}} \phi_{xx} \psi_t \psi_x dx - \int_{\mathbb{T}} \phi_{xt} |\psi_x|^2 dx + \int_{\mathbb{T}} F \psi_t dx.$$

Then, by the Cauchy-Schwarz inequality, the Sobolev imbedding  $H^1 \hookrightarrow L^\infty$  and the estimate

$$\|\mathbb{H}[\varphi]\|_{H^s} \leq \|\varphi\|_{H^s}, \quad \forall \varphi \in H^s, s \in \mathbb{R}, \quad (19)$$

we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|\psi_t\|^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\psi_x|^2 dx \right) \leq C (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3}) (\|\psi_t\|^2 + \|\psi_x\|^2) + \|F\| \|\psi_t\|. \quad (20)$$

From (17) we obtain the estimate

$$\|\psi_x\| \leq \frac{1}{\sqrt{\delta}} \left( \int_{\mathbb{T}} (\mu - 2\phi_x) |\psi_x|^2 dx \right)^{1/2} \leq \frac{1}{\sqrt{\delta}} \mathcal{E}, \quad (21)$$

and, substituting it in (20), we obtain the bound

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\psi_t\|^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\psi_x|^2 dx \right) \\ \leq C (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3}) \left( \|\psi_t\|^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\psi_x|^2 dx \right) + \|F\| \|\psi_t\|, \end{aligned}$$

with a new constant  $C$  also depending on  $\delta$ . Since  $\|\psi_t\| \leq \mathcal{E}$ , the above inequality implies

$$\frac{d}{dt} \mathcal{E} \leq C (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3}) \mathcal{E} + \|F\|, \quad (22)$$

that is (18). Applying the Gronwall lemma to (22) gives the a priori estimate

$$\mathcal{E}(t) \leq e^{CT \max_{\tau \in I} (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3})} \left\{ \mathcal{E}(0) + \int_0^t \|F\| d\tau \right\},$$

which yields

$$\begin{aligned} & (\|\psi_t(t)\|^2 + \|\psi_x(t)\|^2)^{1/2} \\ & \leq C e^{CT \max_{\tau \in I} (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3})} \left\{ \left( \|\psi^{(1)}\|^2 + (\mu + 2\|\varphi^{(0)}\|_{H^2}) \|\psi_x^{(0)}\|^2 \right)^{1/2} + \int_0^T \|F\| d\tau \right\}, \end{aligned} \quad (23)$$

for all  $t \in I$ , with  $C$  also depending on  $\delta$ . Using (23), the existence of one solution is obtained by standard arguments, e.g. see [6]. By linearity of the problem, the difference of any two solutions with the same data satisfies (23) with zero right-hand side. This shows that the solution is defined up to an additive constant. Thus, requiring the solution to have zero mean gives one uniquely defined solution. Since such a solution has zero mean, we can apply the Poincaré inequality (15) and obtain from (23)

$$\begin{aligned} & (\|\psi_t(t)\|^2 + \|\psi(t)\|_{H^1}^2)^{1/2} \\ & \leq C e^{CT \max_{\tau \in I} (\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3})} \left\{ \left( \|\psi^{(1)}\|^2 + (\mu + 2\|\varphi^{(0)}\|_{H^2}) \|\psi_x^{(0)}\|^2 \right)^{1/2} + \int_0^T \|F\| d\tau \right\}. \end{aligned}$$

for all  $t \in I$ , which shows that  $\psi \in C(I; H^1) \cap C^1(I; L^2)$ .  $\square$

The next lemma concerns the regularity of the solution  $\psi$  to (16).

**Lemma 4.** *Let  $r \geq 2$  and  $s \geq \max\{3, r\}$ . Let  $\varphi \in C(I; H^s) \cap C^1(I; H^2)$  satisfying (17). For all  $\psi^{(0)} \in H^r$ ,  $\psi^{(1)} \in H^{r-1}$  with zero mean, and  $F \in L^2(I; H^{r-1})$  there exists a unique solution  $\psi \in C(I; H^r) \cap C^1(I; H^{r-1})$  of (16), with zero mean and such that*

$$\begin{aligned} & \|\psi(t)\|_{H^r}^2 + \|\psi_t(t)\|_{H^{r-1}}^2 \\ & \leq C e^{CT \max_{\tau \in I} (1 + \|\varphi_t\|_{H^2} + \|\varphi\|_{H^s})} \left\{ \|\psi^{(1)}\|_{H^{r-1}}^2 + (\mu + 2\|\varphi^{(0)}\|_{H^2}) \|\psi^{(0)}\|_{H^r}^2 + \|F\|_{L^2(0,t; H^{r-1})}^2 \right\} \end{aligned} \quad (24)$$

for every  $t \in I$ .

*Proof.* We apply  $\langle \partial_x \rangle^{r-1}$  to the equation in (16), multiply by  $\langle \partial_x \rangle^{r-1} \psi_t$  and integrate over  $\mathbb{T}$ . Integrating by parts gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\psi_t\|_{H^{r-1}}^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\langle \partial_x \rangle^{r-1} \psi_x|^2 dx \right) \\ & = 2 \int_{\mathbb{T}} \phi_{xx} \langle \partial_x \rangle^{r-1} \psi_t \langle \partial_x \rangle^{r-1} \psi_x dx - \int_{\mathbb{T}} \phi_{xt} |\langle \partial_x \rangle^{r-1} \psi_x|^2 dx \\ & \quad - 2 \int_{\mathbb{T}} [\langle \partial_x \rangle^{r-1}; \phi_x] \psi_{xx} \langle \partial_x \rangle^{r-1} \psi_t dx + \int_{\mathbb{T}} \langle \partial_x \rangle^{r-1} F \langle \partial_x \rangle^{r-1} \psi_t dx \\ & = \sum_{k=1}^4 I_k. \end{aligned} \quad (25)$$

We estimate each term of this sum.

*Estimate of  $I_1$ .* We apply the Cauchy-Schwarz inequality, the Sobolev imbedding  $H^1 \hookrightarrow L^\infty$  and the estimate (19):

$$|I_1| \leq 2 \|\phi_{xx}\|_{L^\infty} \|\langle \partial_x \rangle^{r-1} \psi_t\| \|\langle \partial_x \rangle^{r-1} \psi_x\| \leq C \|\varphi_x\|_{H^2} \|\psi_t\|_{H^{r-1}} \|\psi_x\|_{H^{r-1}}. \quad (26)$$

*Estimate of  $I_2$ .* In a similar way we obtain

$$|I_2| \leq \|\phi_{xt}\|_{L^\infty} \|\langle \partial_x \rangle^{r-1} \psi_x\|^2 \leq C \|\varphi_t\|_{H^2} \|\psi_x\|_{H^{r-1}}^2. \quad (27)$$

*Estimate of  $I_3$ .* First of all we write

$$|I_3| \leq 2 \|[\langle \partial_x \rangle^{r-1}; \phi_x] \psi_{xx}\| \|\langle \partial_x \rangle^{r-1} \psi_t\|. \quad (28)$$

For the estimate of the commutator we need to distinguish between the different values of  $r$ .

i) If  $r > 5/2$  we apply the estimate (69) for commutators with  $\tau = r - 1, \sigma = r - 2 > 1/2$  and (19) to obtain

$$\begin{aligned} \|[\langle \partial_x \rangle^{r-1}; \phi_x] \psi_{xx}\| &\leq C (\|\phi_x\|_{H^{r-1}} + \|\phi_{xx}\|_{H^1}) \|\psi_{xx}\|_{H^{r-2}} \\ &\leq C (\|\varphi_x\|_{H^{r-1}} + \|\varphi_x\|_{H^2}) \|\psi_x\|_{H^{r-1}} \leq C \|\varphi_x\|_{H^{s-1}} \|\psi_x\|_{H^{r-1}}. \end{aligned} \quad (29)$$

ii) If  $2 \leq r < 5/2$  we apply (70) with  $\tau = r - 1, \sigma = 3 - r > 1/2$  and (19) to get

$$\begin{aligned} \|[\langle \partial_x \rangle^{r-1}; \phi_x] \psi_{xx}\| &\leq C (\|\phi_x\|_{H^2} \|\psi_{xx}\| + \|\phi_{xx}\|_{H^1} \|\psi_{xx}\|_{H^{r-2}}) \\ &\leq C \|\varphi_x\|_{H^2} (\|\psi_x\|_{H^1} + \|\psi_x\|_{H^{r-1}}) \leq C \|\varphi_x\|_{H^{s-1}} \|\psi_x\|_{H^{r-1}}. \end{aligned} \quad (30)$$

iii) Finally if  $r = 5/2$  we apply (71) with  $\tau = r - 1$  and (19) to obtain

$$\begin{aligned} \|[\langle \partial_x \rangle^{r-1}; \phi_x] \psi_{xx}\| &\leq C (\|\phi_x\|_{H^2} \|\psi_{xx}\|_{H^{1/2}} + \|\phi_{xx}\|_{H^1} \|\psi_{xx}\|_{H^{r-2}}) \\ &\leq C \|\varphi_x\|_{H^2} \|\psi_x\|_{H^{3/2}} \leq C \|\varphi_x\|_{H^{s-1}} \|\psi_x\|_{H^{r-1}}. \end{aligned} \quad (31)$$

Recalling (28), from (29)–(31) we have obtained, for all values of  $r \geq 2$ ,

$$|I_3| \leq C \|\varphi_x\|_{H^{s-1}} \|\psi_x\|_{H^{r-1}} \|\psi_t\|_{H^{r-1}}. \quad (32)$$

*Estimate of  $I_4$ .* The Cauchy-Schwarz inequality gives

$$|I_4| \leq \| \langle \partial_x \rangle^{r-1} F \| \| \langle \partial_x \rangle^{r-1} \psi_t \| = \| F \|_{H^{r-1}} \|\psi_t\|_{H^{r-1}}. \quad (33)$$

Estimating the right-hand side of (25) by (26), (27), (32), (33) yields

$$\begin{aligned} \frac{d}{dt} \left( \|\psi_t\|_{H^{r-1}}^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\langle \partial_x \rangle^{r-1} \psi_x|^2 dx \right) \\ \leq C (\|\varphi_t\|_{H^2} + \|\varphi_x\|_{H^{s-1}}) (\|\psi_t\|_{H^{r-1}}^2 + \|\psi_x\|_{H^{r-1}}^2) + 2\|F\|_{H^{r-1}} \|\psi_t\|_{H^{r-1}} \\ \leq C (1 + \|\varphi_t\|_{H^2} + \|\varphi_x\|_{H^{s-1}}) (\|\psi_t\|_{H^{r-1}}^2 + \|\psi_x\|_{H^{r-1}}^2) + \|F\|_{H^{r-1}}^2, \\ \leq C (1 + \|\varphi_t\|_{H^2} + \|\varphi_x\|_{H^{s-1}}) \left( \|\psi_t\|_{H^{r-1}}^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\langle \partial_x \rangle^{r-1} \psi_x|^2 dx \right) + \|F\|_{H^{r-1}}^2, \end{aligned} \quad (34)$$

where in the last inequality we have used (17). Applying the Gronwall lemma to (34) gives

$$\begin{aligned} \|\psi_t(t)\|_{H^{r-1}}^2 + \int_{\mathbb{T}} (\mu - 2\phi_x) |\langle \partial_x \rangle^{r-1} \psi_x(t)|^2 dx \\ \leq e^{CT \max_{\tau \in I} (1 + \|\varphi_t\|_{H^2} + \|\varphi_x\|_{H^{s-1}})} \left\{ \|\psi^{(1)}\|_{H^{r-1}}^2 + \int_{\mathbb{T}} (\mu - 2\phi_x(0)) |\langle \partial_x \rangle^{r-1} \psi_x^{(0)}|^2 dx + \int_0^t \|F\|_{H^{r-1}}^2 d\tau \right\}, \end{aligned}$$

which yields, by (17) and the Poincaré inequality,

$$\begin{aligned} \|\psi_t(t)\|_{H^{r-1}}^2 + \|\psi(t)\|_{H^r}^2 \\ \leq C e^{CT \max_{\tau \in I} (1 + \|\varphi_t\|_{H^2} + \|\varphi\|_{H^s})} \left\{ \|\psi^{(1)}\|_{H^{r-1}}^2 + (\mu + 2\|\varphi^{(0)}\|_{H^2}) \|\psi^{(0)}\|_{H^r}^2 + \int_0^t \|F\|_{H^{r-1}}^2 d\tau \right\}, \end{aligned} \quad (35)$$

for all  $t \in I$ , with  $C$  also depending on  $\delta$ , that is (24). (35) provides the a priori estimate for  $\psi$  in  $C(I; H^r) \cap C^1(I; H^{r-1})$ .  $\square$

Now we consider the quadratic operator  $\mathcal{Q}[\varphi]$ , defined in (4). First of all we recall the estimate proved in [7].

**Proposition 5.** *There exists a positive constant  $C$  such that for every real  $s \geq 1$  and for all  $\varphi \in H^s \cap H^3$*

$$\|\mathcal{Q}[\varphi]\|_{H^{s-1}} \leq C \|\varphi_x\|_{H^2} \|\varphi_x\|_{H^{s-1}}. \quad (36)$$

*Proof.* For the proof see [7].  $\square$

Next we give an estimate for the difference of values of  $\mathcal{Q}[\varphi]$ .

**Lemma 6.** *The quadratic operator  $\mathcal{Q}[\varphi]$ , defined in (4), satisfies*

$$\|\mathcal{Q}[\varphi] - \mathcal{Q}[\tilde{\varphi}]\| \leq C (\|\varphi\|_{H^3} + \|\tilde{\varphi}\|_{H^3}) \|(\varphi - \tilde{\varphi})_x\| \quad (37)$$

for all functions  $\varphi, \tilde{\varphi} \in H^3$ .

*Proof.* We denote  $\phi = \mathbb{H}[\varphi]$ ,  $\tilde{\phi} = \mathbb{H}[\tilde{\varphi}]$ ,  $\delta\varphi = \varphi - \tilde{\varphi}$ ,  $\delta\phi = \phi - \tilde{\phi}$ . Then we can write

$$\begin{aligned} \mathcal{Q}[\varphi] - \mathcal{Q}[\tilde{\varphi}] &= -3[\mathbb{H}; \phi_x] \phi_{xx} - [\mathbb{H}; \phi] \phi_{xxx} + 3[\mathbb{H}; \tilde{\phi}_x] \tilde{\phi}_{xx} + [\mathbb{H}; \tilde{\phi}] \tilde{\phi}_{xxx} \\ &= -3[\mathbb{H}; \delta\phi_x] \phi_{xx} - 3[\mathbb{H}; \tilde{\phi}_x] \delta\phi_{xx} - [\mathbb{H}; \delta\phi] \phi_{xxx} - [\mathbb{H}; \tilde{\phi}] \delta\phi_{xxx} \\ &= \sum_{k=1}^4 J_k. \end{aligned}$$

Now we estimate each term of the sum.

*Estimate of  $J_1$ .* We apply estimate (68), with  $p = s = 0$ , and (19):

$$\|J_1\| \leq C \|\delta\phi_x\| \|\phi_{xx}\|_{H^1} \leq C \|\delta\varphi_x\| \|\varphi\|_{H^3}. \quad (38)$$

*Estimate of  $J_2$ .* We apply estimate (68), with  $p = 2, s = 0$ , (19) and the Poincaré inequality:

$$\|J_2\| \leq C \|\tilde{\phi}_{xxx}\| \|\delta\phi\|_{H^1} \leq C \|\tilde{\varphi}\|_{H^3} \|\delta\varphi_x\|. \quad (39)$$

*Estimate of  $J_3$ .* We apply estimate (67), with  $s = 1$ , (19) and the Poincaré inequality:

$$\|J_3\| \leq C \|\delta\phi\|_{H^1} \|\phi_{xxx}\| \leq C \|\delta\varphi_x\| \|\varphi\|_{H^3}. \quad (40)$$

*Estimate of  $J_4$ .* We apply estimate (68), with  $p = 3, s = 0$ , (19) and the Poincaré inequality:

$$\|J_4\| \leq C \|\tilde{\phi}_{xxx}\| \|\delta\phi\|_{H^1} \leq C \|\tilde{\varphi}\|_{H^3} \|\delta\varphi_x\|. \quad (41)$$

Collecting (38)–(41) gives (37).  $\square$

More generally we apply (68) with different choices of  $p$  to estimate the  $H^{s-1}$ -norm of  $J_1 - J_4$  and obtain

**Lemma 7.** *Let  $s \geq 3$ . The quadratic operator  $\mathcal{Q}[\varphi]$ , defined in (4), satisfies*

$$\|\mathcal{Q}[\varphi] - \mathcal{Q}[\tilde{\varphi}]\|_{H^{s-1}} \leq C (\|\varphi\|_{H^s} + \|\tilde{\varphi}\|_{H^s}) \|\varphi - \tilde{\varphi}\|_{H^s} \quad (42)$$

for all functions  $\varphi, \tilde{\varphi} \in H^s$ .

### 3. PROOF OF THEOREM 2

By assumption the sequence  $\{\varphi_n^{(0)}\}_{n \in \mathbb{N}}$  is uniformly bounded in  $H^s$ , and the sequence  $\{\varphi_n^{(1)}\}_{n \in \mathbb{N}}$  is uniformly bounded in  $H^{s-1}$ . Because of the uniform a priori estimate (10), from now on we may assume that the sequence of solutions  $\{\varphi_n\}_{n \in \mathbb{N}}$  is uniformly bounded in  $C(I; H^s) \cap C^1(I; H^{s-1})$  on the common time interval  $I = [0, T]$ , i.e. there exists  $K > 0$  such that

$$\|\varphi_n(t)\|_{H^s}^2 + \|(\varphi_n)_t(t)\|_{H^{s-1}}^2 \leq C_2 \left( \|\varphi_n^{(0)}\|_{H^s}^2 + \|\varphi_n^{(1)}\|_{H^{s-1}}^2 \right) \leq K, \quad \forall t \in I, \forall n. \quad (43)$$

Notice that the similar bound holds as well for the solution  $\varphi$ .

From now on  $T$  will be usually included in the generic constant  $C$ , but sometimes not, when we prefer to emphasize its presence.

In the next proposition we prove the convergence of  $\{\varphi_n\}_{n \in \mathbb{N}}$  in a weaker topology than in our main Theorem 2.

**Proposition 8.** *Let  $s \geq 3$ . Under the assumptions of Theorem 2 the sequence of solutions  $\{\varphi_n\}_{n \in \mathbb{N}}$  converges to  $\varphi$  strongly in  $C(I; H^{s-1}) \cap C^1(I; H^{s-2})$ .*

*Proof.* We take the difference of equation (3) for  $\varphi$  and  $\varphi_n$  and get

$$(\varphi - \varphi_n)_{tt} - (\mu - 2\phi_x)(\varphi - \varphi_n)_{xx} = -\mathcal{Q}[\varphi] + \mathcal{Q}[\varphi_n] + 2(\phi_{n,x} - \phi_x)\varphi_{n,xx},$$

(where  $\phi_{n,x} = (\phi_n)_x$ ,  $\varphi_{n,xx} = (\varphi_n)_{xx}$ ) which has the form of (16) with

$$\psi = \varphi - \varphi_n,$$

$$F = -\mathcal{Q}[\varphi] + \mathcal{Q}[\varphi_n] + 2(\phi_{n,x} - \phi_x)\varphi_{n,xx}.$$

Applying (37) gives

$$\|\mathcal{Q}[\varphi] - \mathcal{Q}[\varphi_n]\| \leq C (\|\varphi\|_{H^3} + \|\varphi_n\|_{H^3}) \|\psi_x\|. \quad (44)$$

We also have

$$\|2(\phi_{n,x} - \phi_x) \varphi_{n,xx}\| \leq 2\|\phi_{n,x} - \phi_x\| \|\varphi_{n,xx}\|_{L^\infty} \leq C\|\psi_x\| \|\varphi_n\|_{H^3}. \quad (45)$$

Thus, from (44), (45) we obtain

$$\|F\| \leq C(\|\varphi\|_{H^3} + \|\varphi_n\|_{H^3}) \|\psi_x\|. \quad (46)$$

From Lemma 3, (21), (46) we get

$$\frac{d}{dt} \mathcal{E} \leq C(\|\varphi_t\|_{H^2} + \|\varphi\|_{H^3}) \mathcal{E} + C(\|\varphi\|_{H^3} + \|\varphi_n\|_{H^3}) \|\psi_x\| \leq C\mathcal{E}, \quad (47)$$

where we have used the uniform boundedness (43) for  $\varphi_n$  and  $\varphi$ . Applying Gronwall's lemma to (47) and using (9) again yields

$$\|\psi_t(t)\|^2 + \|\psi_x(t)\|^2 \leq Ce^{CT} (\|\psi_t(0)\|^2 + \|\psi_x(0)\|^2) \quad t \in I,$$

that is

$$\|(\varphi - \varphi_n)_t(t)\|^2 + \|(\varphi - \varphi_n)_x(t)\|^2 \leq Ce^{CT} (\|\varphi^{(1)} - \varphi_n^{(1)}\|^2 + \|(\varphi^{(0)} - \varphi_n^{(0)})_x\|^2) \quad t \in I,$$

which gives the strong convergence of  $\varphi_n$  to  $\varphi$  in  $C(I; H^1) \cap C^1(I; L^2)$ , when passing to the limit as  $n \rightarrow +\infty$ . Recall that, since we are working with functions with zero spatial mean, the Poincaré inequality (15) holds. By interpolation and the uniform boundedness (43) we get

$$\begin{aligned} & \|(\varphi - \varphi_n)_t(t)\|_{H^{s-2}}^2 + \|(\varphi - \varphi_n)_t(t)\|_{H^{s-1}}^2 \\ & \leq C \left( \|(\varphi - \varphi_n)_t(t)\|_{H^{s-1}}^{1-1/(s-1)} \|(\varphi - \varphi_n)_t(t)\|_{L^2}^{1/(s-1)} + \|(\varphi - \varphi_n)_t(t)\|_{H^s}^{1-1/(s-1)} \|(\varphi - \varphi_n)_t(t)\|_{H^1}^{1/(s-1)} \right)^2 \\ & \leq C \left( \|(\varphi - \varphi_n)_t(t)\|_{L^2}^{2/(s-1)} + \|(\varphi - \varphi_n)_t(t)\|_{H^1}^{2/(s-1)} \right), \quad t \in I. \end{aligned}$$

Finally, passing to the limit as  $n \rightarrow +\infty$  in the above inequality gives the thesis.  $\square$

**Remark 9.** Obviously, by a similar argument with a finer interpolation we could prove the strong convergence of  $\varphi_n$  to  $\varphi$  in  $C(I; H^{s-\varepsilon}) \cap C^1(I; H^{s-1-\varepsilon})$ , for all small enough  $\varepsilon > 0$ . However, this is useless for the following argument.

Now we take one spatial derivative of (3) and get

$$(\varphi_x)_{tt} - (\mu - 2\phi_x)(\varphi_x)_{xx} = -\mathcal{Q}[\varphi]_x - 2\phi_{xx}\varphi_{xx},$$

which has the form of (16) with

$$\psi = \varphi_x, \quad F = -\mathcal{Q}[\varphi]_x - 2\phi_{xx}\varphi_{xx}.$$

Using Proposition 5, a Moser-type estimate, the Sobolev imbedding and (19) we compute

$$\begin{aligned} \|F\|_{H^{s-2}} & \leq \|\mathcal{Q}[\varphi]\|_{H^{s-1}} + 2\|\phi_{xx}\varphi_{xx}\|_{H^{s-2}} \\ & \leq C(\|\varphi_x\|_{H^2}\|\varphi_x\|_{H^{s-1}} + \|\phi_{xx}\|_{L^\infty}\|\varphi_{xx}\|_{H^{s-2}} + \|\phi_{xx}\|_{H^{s-2}}\|\varphi_{xx}\|_{L^\infty}) \\ & \leq C\|\varphi\|_{H^3}\|\varphi\|_{H^s}, \end{aligned} \quad (48)$$

and we deduce that  $F \in L^\infty(I; H^{s-2})$ . Moreover, the initial values of  $\varphi_x$ ,  $(\varphi_x)_t$  are  $\varphi_x^{(0)} \in H^{s-1}$ ,  $\varphi_x^{(1)} \in H^{s-2}$ , respectively.

We are going to introduce regularized approximations of the data  $\varphi_x^{(0)}$ ,  $\varphi_x^{(1)}$ ,  $F$ . Given any  $\varepsilon > 0$ , let us take functions  $\Psi_\varepsilon^{(0)} \in H^s$ ,  $\Psi_\varepsilon^{(1)} \in H^{s-1}$  with zero mean, and  $F^\varepsilon \in L^2(I; H^{s-1})$  such that

$$\|\Psi_\varepsilon^{(0)} - \varphi_x^{(0)}\|_{H^{s-1}} + \|\Psi_\varepsilon^{(1)} - \varphi_x^{(1)}\|_{H^{s-2}} + \|F^\varepsilon - F\|_{L^2(I; H^{s-2})} \leq \varepsilon. \quad (49)$$

Let us consider the Cauchy problem with regularized data

$$\begin{cases} \Psi_{tt}^\varepsilon - (\mu - 2\phi_x) \Psi_{xx}^\varepsilon = F^\varepsilon, & t \in I, x \in \mathbb{T}, \\ \Psi|_{t=0}^\varepsilon = \Psi_\varepsilon^{(0)}, \quad \partial_t \Psi|_{t=0}^\varepsilon = \Psi_\varepsilon^{(1)} & x \in \mathbb{T}. \end{cases} \quad (50)$$

Again this problem has the form (16).



**Lemma 10.** *Let  $s \geq 3$ . For any  $\varepsilon > 0$ , the Cauchy problem (50) has a unique solution  $\Psi^\varepsilon \in C(I; H^s) \cap C^1(I; H^{s-1})$  and*

$$\begin{aligned} & \|\Psi^\varepsilon\|_{C(I; H^s)}^2 + \|\Psi_t^\varepsilon\|_{C(I; H^{s-1})}^2 \\ & \leq C e^{CT \max_{\tau \in I} (1 + \|\varphi_\tau\|_{H^2} + \|\varphi\|_{H^s})} \left\{ \|\Psi_\varepsilon^{(1)}\|_{H^{s-1}}^2 + (\mu + 2\|\varphi^{(0)}\|_{H^2}) \|\Psi_\varepsilon^{(0)}\|_{H^s}^2 + \|F^\varepsilon\|_{L^2(I; H^{s-1})}^2 \right\}. \end{aligned} \quad (51)$$

*Proof.* The result follows from Lemma 4 with  $r = s$ .  $\square$

Now we estimate the difference  $\Psi^\varepsilon - \varphi_x$ , which solves the linear problem

$$\begin{cases} (\Psi^\varepsilon - \varphi_x)_{tt} - (\mu - 2\phi_x)(\Psi^\varepsilon - \varphi_x)_{xx} = F^\varepsilon - F, & t \in I, x \in \mathbb{T}, \\ (\Psi^\varepsilon - \varphi_x)|_{t=0} = \Psi_\varepsilon^{(0)} - \varphi_x^{(0)}, \quad \partial_t(\Psi^\varepsilon - \varphi_x)|_{t=0} = \Psi_\varepsilon^{(1)} - \varphi_x^{(1)} & x \in \mathbb{T}. \end{cases} \quad (52)$$

**Lemma 11.** *Let  $s \geq 3$ . For any  $\varepsilon > 0$ , the difference  $\Psi^\varepsilon - \varphi_x$  satisfies the estimate*

$$\|(\Psi^\varepsilon - \varphi_x)(t)\|_{H^{s-1}} + \|(\Psi^\varepsilon - \varphi_x)_t(t)\|_{H^{s-2}} \leq C\varepsilon \quad \forall t \in I. \quad (53)$$

*Proof.* We apply Lemma 4 with  $r = s - 1$  and get

$$\begin{aligned} & \|\Psi^\varepsilon - \varphi_x\|_{C(I; H^{s-1})}^2 + \|(\Psi^\varepsilon - \varphi_x)_t\|_{C(I; H^{s-2})}^2 \\ & \leq C e^{CT \max_{\tau \in I} (1 + \|\varphi_\tau\|_{H^2} + \|\varphi\|_{H^s})} \left\{ \|\Psi_\varepsilon^{(1)} - \varphi_x^{(1)}\|_{H^{s-2}}^2 \right. \\ & \quad \left. + (\mu + 2\|\varphi^{(0)}\|_{H^2}) \|\Psi_\varepsilon^{(0)} - \varphi_x^{(0)}\|_{H^{s-1}}^2 + \|F^\varepsilon - F\|_{L^2(I; H^{s-2})}^2 \right\}. \end{aligned} \quad (54)$$

Thus (53) follows from (49).  $\square$

Finally we estimate the difference between  $\varphi_{n,x} = (\varphi_n)_x$  and  $\Psi^\varepsilon$ . The difference of the corresponding problems reads

$$\begin{cases} (\Psi^\varepsilon - \varphi_{n,x})_{tt} - (\mu - 2\phi_{n,x})(\Psi^\varepsilon - \varphi_{n,x})_{xx} = G^{n,\varepsilon}, & t \in I, x \in \mathbb{T}, \\ (\Psi^\varepsilon - \varphi_{n,x})|_{t=0} = \Psi_\varepsilon^{(0)} - \varphi_{n,x}^{(0)}, \\ \partial_t(\Psi^\varepsilon - \varphi_{n,x})|_{t=0} = \Psi_\varepsilon^{(1)} - \varphi_{n,x}^{(1)} & x \in \mathbb{T}. \end{cases} \quad (55)$$

where we have set

$$\begin{aligned} G^{n,\varepsilon} &= F^\varepsilon - F^n + 2(\phi_{n,x} - \phi_x) \Psi_{xx}^\varepsilon, \quad F^n = -\mathcal{Q}[\varphi_n]_x - 2\phi_{n,xx} \varphi_{n,xx}, \\ \varphi_{n,x}^{(0)} &= (\varphi_n^{(0)})_x, \quad \varphi_{n,x}^{(1)} = (\varphi_n^{(1)})_x. \end{aligned} \quad (56)$$

**Lemma 12.** *Let  $s \geq 3$ . For any  $\varepsilon > 0$ , there exists  $M(\varepsilon) > 0$  such that, for any  $n$  the difference  $\Psi^\varepsilon - \varphi_{n,x}$  satisfies the estimate*

$$\begin{aligned} & \|(\Psi^\varepsilon - \varphi_{n,x})(t)\|_{H^{s-1}}^2 + \|(\Psi^\varepsilon - \varphi_{n,x})_t(t)\|_{H^{s-2}}^2 \\ & \leq C \left\{ \varepsilon^2 + \|\varphi^{(1)} - \varphi_n^{(1)}\|_{H^{s-1}}^2 + \|\varphi^{(0)} - \varphi_n^{(0)}\|_{H^s}^2 + \left| \int_0^t \|\varphi - \varphi_n\|_{H^s}^2 d\tau \right| + TM(\varepsilon) \|\varphi_n - \varphi\|_{C(I; H^{s-1})}^2 \right\} \end{aligned} \quad (57)$$

for any  $t \in I$ .

*Proof.* We apply Lemma 4 with  $r = s - 1$  and obtain for every  $t \in I$

$$\begin{aligned} & \|(\Psi^\varepsilon - \varphi_{n,x})(t)\|_{H^{s-1}}^2 + \|(\Psi^\varepsilon - \varphi_{n,x})_t(t)\|_{H^{s-2}}^2 \\ & \leq C e^{CT \max_{\tau \in I} (1 + \|\varphi_{n,\tau}\|_{H^2} + \|\varphi_n\|_{H^s})} \left\{ \|\Psi_\varepsilon^{(1)} - \varphi_{n,x}^{(1)}\|_{H^{s-2}}^2 \right. \\ & \quad \left. + (\mu + 2\|\varphi_n^{(0)}\|_{H^2}) \|\Psi_\varepsilon^{(0)} - \varphi_{n,x}^{(0)}\|_{H^{s-1}}^2 + \|G^{n,\varepsilon}\|_{L^2(0,t; H^{s-2})}^2 \right\}. \end{aligned} \quad (58)$$

First of all, recalling the uniform boundedness w.r.t.  $n$  (43), we may write (58) as

$$\begin{aligned} & \|(\Psi^\varepsilon - \varphi_{n,x})(t)\|_{H^{s-1}}^2 + \|(\Psi^\varepsilon - \varphi_{n,x})_t(t)\|_{H^{s-2}}^2 \\ & \leq C \left\{ \|\Psi_\varepsilon^{(1)} - \varphi_{n,x}^{(1)}\|_{H^{s-2}}^2 + \|\Psi_\varepsilon^{(0)} - \varphi_{n,x}^{(0)}\|_{H^{s-1}}^2 + \|G^{n,\varepsilon}\|_{L^2(0,t;H^{s-2})}^2 \right\}. \end{aligned} \quad (59)$$

From the definition in (56) we have, for every  $\tau \in [0, t]$ ,

$$\begin{aligned} \|G^{n,\varepsilon}\|_{H^{s-2}} & \leq \|F^\varepsilon - F^n\|_{H^{s-2}} + 2\|(\phi_{n,x} - \phi_x)\Psi_{xx}^\varepsilon\|_{H^{s-2}} \\ & \leq \|F^\varepsilon - F\|_{H^{s-2}} + \|F - F^n\|_{H^{s-2}} + C\|\varphi_n - \varphi\|_{H^{s-1}}\|\Psi^\varepsilon\|_{H^s}. \end{aligned}$$

Integrating this inequality in  $\tau$  between 0 and  $t$  gives

$$\left| \int_0^t \|G^{n,\varepsilon}\|_{H^{s-2}}^2 d\tau \right| \leq 3 \left| \int_0^t \|F^\varepsilon - F\|_{H^{s-2}}^2 d\tau \right| + 3 \left| \int_0^t \|F - F^n\|_{H^{s-2}}^2 d\tau \right| + CTM(\varepsilon)\|\varphi_n - \varphi\|_{C(I;H^{s-1})}^2 \quad (60)$$

where we have denoted

$$M(\varepsilon) := Ce^{CT \max_{\tau \in I} (1 + \|\varphi_t\|_{H^2} + \|\varphi\|_{H^s})} \left\{ \|\Psi_\varepsilon^{(1)}\|_{H^{s-1}}^2 + (\mu + 2\|\varphi^{(0)}\|_{H^2})\|\Psi_\varepsilon^{(0)}\|_{H^s}^2 + \|F^\varepsilon\|_{L^2(I;H^{s-1})}^2 \right\},$$

that is the right-hand side of (51). On the other hand, for all  $\tau$  we have

$$\|F - F^n\|_{H^{s-2}} \leq \|\mathcal{Q}[\varphi] - \mathcal{Q}[\varphi_n]\|_{H^{s-1}} + 2\|\phi_{xx}\varphi_{xx} - \phi_{n,xx}\varphi_{n,xx}\|_{H^{s-2}}. \quad (61)$$

From (42) we have

$$\|\mathcal{Q}[\varphi] - \mathcal{Q}[\varphi_n]\|_{H^{s-1}} \leq C(\|\varphi\|_{H^s} + \|\varphi_n\|_{H^s})\|\varphi - \varphi_n\|_{H^s}. \quad (62)$$

Moreover, since  $H^{s-2}$  is an algebra we can estimate

$$\begin{aligned} & 2\|\phi_{xx}\varphi_{xx} - \phi_{n,xx}\varphi_{n,xx}\|_{H^{s-2}} \\ & \leq C\|\phi_{xx} - \phi_{n,xx}\|_{H^{s-2}}\|\varphi_{xx}\|_{H^{s-2}} + C\|\phi_{n,xx}\|_{H^{s-2}}\|\varphi_{xx} - \varphi_{n,xx}\|_{H^{s-2}} \\ & \leq C(\|\varphi\|_{H^s} + \|\varphi_n\|_{H^s})\|\varphi - \varphi_n\|_{H^s}. \end{aligned} \quad (63)$$

From (61)–(63) and the uniform boundedness (43) we obtain

$$\|F - F^n\|_{H^{s-2}} \leq C\|\varphi - \varphi_n\|_{H^s} \quad \forall \tau \in I, \quad (64)$$

and substituting it in (60) gives

$$\left| \int_0^t \|G^{n,\varepsilon}\|_{H^{s-2}}^2 d\tau \right| \leq 3 \left| \int_0^t \|F^\varepsilon - F\|_{H^{s-2}}^2 d\tau \right| + C \left| \int_0^t \|\varphi - \varphi_n\|_{H^s}^2 d\tau \right| + CTM(\varepsilon)\|\varphi_n - \varphi\|_{C(I;H^{s-1})}^2. \quad (65)$$

Finally, from (49), (59), (65) we get

$$\begin{aligned} & \|(\Psi^\varepsilon - \varphi_{n,x})(t)\|_{H^{s-1}}^2 + \|(\Psi^\varepsilon - \varphi_{n,x})_t(t)\|_{H^{s-2}}^2 \\ & \leq C \left\{ \varepsilon^2 + \|\varphi^{(1)} - \varphi_n^{(1)}\|_{H^{s-1}}^2 + \|\varphi^{(0)} - \varphi_n^{(0)}\|_{H^s}^2 + \left| \int_0^t \|\varphi - \varphi_n\|_{H^s}^2 d\tau \right| + TM(\varepsilon)\|\varphi_n - \varphi\|_{C(I;H^{s-1})}^2 \right\} \end{aligned}$$

for all  $t \in I$ , that is (57).  $\square$

*Proof of Theorem 2.* Adding (53), (57), and applying the Poincaré inequality gives

$$\begin{aligned} & \|(\varphi - \varphi_n)(t)\|_{H^s}^2 + \|(\varphi - \varphi_n)_t(t)\|_{H^{s-1}}^2 \\ & \leq C \left\{ \varepsilon^2 + \|\varphi^{(1)} - \varphi_n^{(1)}\|_{H^{s-1}}^2 + \|\varphi^{(0)} - \varphi_n^{(0)}\|_{H^s}^2 + \left| \int_0^t \|\varphi - \varphi_n\|_{H^s}^2 d\tau \right| + TM(\varepsilon)\|\varphi_n - \varphi\|_{C(I;H^{s-1})}^2 \right\} \end{aligned}$$

for all  $t \in I$ . Then, applying the Gronwall lemma yields

$$\begin{aligned} & \|\varphi - \varphi_n\|_{C(I; H^s)}^2 + \|(\varphi - \varphi_n)_t\|_{C(I; H^{s-1})}^2 \\ & \leq C_3 \left\{ \varepsilon^2 + \|\varphi^{(1)} - \varphi_n^{(1)}\|_{H^{s-1}}^2 + \|\varphi^{(0)} - \varphi_n^{(0)}\|_{H^s}^2 + M(\varepsilon) \|\varphi_n - \varphi\|_{C(I; H^{s-1})}^2 \right\}. \end{aligned} \quad (66)$$

Given any  $\varepsilon' > 0$ , let  $\varepsilon = \varepsilon(\varepsilon')$  be such that  $C_3 \varepsilon^2 < \varepsilon'/3$ . With this fixed  $\varepsilon$  in  $M(\varepsilon)$ , and taking account of Proposition 8, let  $n_0$  be such that, for any  $n \geq n_0$ ,

$$\begin{aligned} C_3 \left\{ \|\varphi^{(1)} - \varphi_n^{(1)}\|_{H^{s-1}}^2 + \|\varphi^{(0)} - \varphi_n^{(0)}\|_{H^s}^2 \right\} & < \varepsilon'/3, \\ C_3 M(\varepsilon) \|\varphi_n - \varphi\|_{C(I; H^{s-1})}^2 & < \varepsilon'/3. \end{aligned}$$

It follows from (66) that

$$\|\varphi - \varphi_n\|_{C(I; H^s)}^2 + \|(\varphi - \varphi_n)_t\|_{C(I; H^{s-1})}^2 < \varepsilon' \quad \forall n \geq n_0.$$

This concludes the proof of Theorem 2.  $\square$

#### APPENDIX A. SOME COMMUTATOR ESTIMATES

**Lemma 13.** *For  $\tau > 1/2$  there exists a constant  $C_\tau > 0$  such that*

$$\|[\mathbb{H}; v] f\|_{L^2(\mathbb{T})} \leq C_\tau \|v\|_{H^\tau(\mathbb{T})} \|f\|_{L^2(\mathbb{T})}, \quad \forall v \in H^\tau(\mathbb{T}), \forall f \in L^2(\mathbb{T}), \quad (67)$$

where  $[\mathbb{H}; v]$  is the commutator between the Hilbert transform  $\mathbb{H}$  and the multiplication by  $v$ .

*Proof.* The proof can be found in [8].  $\square$

**Lemma 14.** *For every real  $\tau \geq 0$  and integer  $p \geq 0$  there exists a constant  $C_{\tau, p} > 0$  such that for all functions  $v \in H^{\tau+p}(\mathbb{T})$  and  $f \in H^1(\mathbb{T})$*

$$\|[\mathbb{H}; v] \partial_x^p f\|_{H^\tau(\mathbb{T})} \leq C_{\tau, p} \|\partial_x^p v\|_{H^\tau(\mathbb{T})} \|f\|_{H^1(\mathbb{T})}. \quad (68)$$

*Proof.* The proof can be found in [7].  $\square$

**Lemma 15.** *For every real  $\tau \geq 1$  and  $\sigma > 1/2$  there exists a constant  $C_{\tau, \sigma} > 0$  such that*

$$\begin{aligned} \text{i. for all } f \in H^{\tau-1}(\mathbb{T}) \cap H^\sigma(\mathbb{T}) \text{ and } v \in H^\tau(\mathbb{T}) \cap H^2(\mathbb{T}) \\ \|[\langle \partial_x \rangle^\tau; v] f\|_{L^2(\mathbb{T})} \leq C_{\tau, \sigma} \left\{ \|v\|_{H^\tau(\mathbb{T})} \|f\|_{H^\sigma(\mathbb{T})} + \|v_x\|_{H^1(\mathbb{T})} \|f\|_{H^{\tau-1}(\mathbb{T})} \right\}; \end{aligned} \quad (69)$$

$$\begin{aligned} \text{ii. for all } f \in H^{\tau-1}(\mathbb{T}) \text{ and } v \in H^{\tau+\sigma}(\mathbb{T}) \cap H^2(\mathbb{T}) \\ \|[\langle \partial_x \rangle^\tau; v] f\|_{L^2(\mathbb{T})} \leq C_{\tau, \sigma} \left\{ \|v\|_{H^{\tau+\sigma}(\mathbb{T})} \|f\|_{L^2(\mathbb{T})} + \|v_x\|_{H^1(\mathbb{T})} \|f\|_{H^{\tau-1}(\mathbb{T})} \right\}. \end{aligned} \quad (70)$$

For all  $\tau \geq 1$  there exists a positive constant  $C_\tau$  such that for all  $f \in H^{\tau-1}(\mathbb{T}) \cap H^{1/2}(\mathbb{T})$  and  $v \in H^{\tau+1/2}(\mathbb{T}) \cap H^2(\mathbb{T})$

$$\|[\langle \partial_x \rangle^\tau; v] f\|_{L^2(\mathbb{T})} \leq C_\tau \left\{ \|v\|_{H^{\tau+1/2}(\mathbb{T})} \|f\|_{H^{1/2}(\mathbb{T})} + \|v_x\|_{H^1(\mathbb{T})} \|f\|_{H^{\tau-1}(\mathbb{T})} \right\}. \quad (71)$$

*Proof.* For all  $k \in \mathbb{Z}$  we compute

$$\begin{aligned} [\widehat{\langle \partial_x \rangle^\tau}; v] f(k) &= \langle k \rangle^\tau \widehat{v f}(k) - v \widehat{\langle \partial_x \rangle^\tau f}(k) \\ &= \frac{1}{2\pi} \langle k \rangle^\tau \sum_\ell \widehat{v}(k-\ell) \widehat{f}(\ell) - \frac{1}{2\pi} \sum_\ell \widehat{v}(k-\ell) \langle \ell \rangle^\tau \widehat{f}(\ell) \\ &= \frac{1}{2\pi} \sum_\ell (\langle k \rangle^\tau - \langle \ell \rangle^\tau) \widehat{v}(k-\ell) \widehat{f}(\ell). \end{aligned} \quad (72)$$

On the other hand we have

$$\langle k \rangle^\tau - \langle \ell \rangle^\tau = \int_0^1 \frac{d}{d\theta} (\langle \ell + \theta(k-\ell) \rangle^\tau) d\theta = (k-\ell) \int_0^1 D(\langle \cdot \rangle^\tau) (\ell + \theta(k-\ell)) d\theta,$$

where  $D$  denotes the derivative of the function  $\langle \cdot \rangle^\tau$ . Combining the preceding with the estimate

$$\left| \frac{d}{d\xi} \langle \xi \rangle^\tau \right| \leq C_\tau \langle \xi \rangle^{\tau-1}, \quad \forall \xi \in \mathbb{R}, \quad (73)$$

then gives

$$|\langle k \rangle^\tau - \langle \ell \rangle^\tau| \leq |k - \ell| \int_0^1 |D(\langle \cdot \rangle^\tau)(\ell + \theta(k - \ell))| d\theta \leq C_\tau |k - \ell| \int_0^1 \langle \ell + \theta(k - \ell) \rangle^{\tau-1} d\theta. \quad (74)$$

Using (74), from (72) we get

$$\begin{aligned} |[\langle \partial_x \rangle^\tau; v] f(k)| &\leq \frac{1}{2\pi} \sum_\ell |\langle k \rangle^\tau - \langle \ell \rangle^\tau| |\widehat{v}(k - \ell)| |\widehat{f}(\ell)| \\ &\leq C_\tau \sum_\ell \int_0^1 |k - \ell| \langle \ell + \theta(k - \ell) \rangle^{\tau-1} |\widehat{v}(k - \ell)| |\widehat{f}(\ell)| d\theta. \end{aligned} \quad (75)$$

Since the function  $\langle \zeta \rangle^{\tau-1}$  is sub-additive and  $0 \leq \theta \leq 1$ , we have

$$\langle \ell + \theta(k - \ell) \rangle^{\tau-1} \leq C_\tau \{ \langle \theta(k - \ell) \rangle^{\tau-1} + \langle \ell \rangle^{\tau-1} \} \leq C_\tau \{ \langle k - \ell \rangle^{\tau-1} + \langle \ell \rangle^{\tau-1} \}, \quad (76)$$

with positive constant  $C_\tau$  depending only on  $\tau$ . Using (76) to estimate the right-hand side of (75) then gives

$$\begin{aligned} |[\langle \partial_x \rangle^\tau; v] f(k)| &\leq C_\tau \sum_\ell \left\{ \int_0^1 \langle k - \ell \rangle^\tau |\widehat{v}(k - \ell)| |\widehat{f}(\ell)| d\theta + \int_0^1 |k - \ell| |\widehat{v}(k - \ell)| \langle \ell \rangle^{\tau-1} |\widehat{f}(\ell)| d\theta \right\} \\ &\leq C'_\tau \left\{ \left( |\widehat{\langle \partial_x \rangle^\tau v}| * |\widehat{f}| \right)(k) + \left( |\widehat{v}_x| * |\widehat{\langle \partial_x \rangle^{\tau-1} f}| \right)(k) \right\}. \end{aligned} \quad (77)$$

Using Parseval's identity and Young's inequality with  $\{|\widehat{\langle \partial_x \rangle^\tau v}(k)|\} \in \ell^2$ ,  $\{|\widehat{f}(k)|\} \in \ell^1$ ,  $\{|\widehat{v}_x(k)|\} \in \ell^1$ ,  $\{|\widehat{\langle \partial_x \rangle^{\tau-1} f}(k)|\} \in \ell^2$ , from (77) we derive

$$\|[\langle \partial_x \rangle^\tau; v] f\|_{L^2(\mathbb{T})} \leq C'_\tau \left\{ \|\{|\widehat{\langle \partial_x \rangle^\tau v}|\}\|_{\ell^2} \|\{|\widehat{f}|\}\|_{\ell^1} + \|\{|\widehat{v}_x|\}\|_{\ell^1} \|\{|\widehat{\langle \partial_x \rangle^{\tau-1} f}|\}\|_{\ell^2} \right\}. \quad (78)$$

We get the first inequality (69) of Lemma 15, by using once again Parseval's identity and the estimates

$$\|\{|\widehat{f}|\}\|_{\ell^1} \leq C_\sigma \|f\|_{H^\sigma(\mathbb{T})}, \quad \|\{|\widehat{v}_x|\}\|_{\ell^1} \leq C \|v_x\|_{H^1(\mathbb{T})}. \quad (79)$$

To get the second inequality (70) it is sufficient to interchange the role of the sequences  $\{|\widehat{\langle \partial_x \rangle^\tau v}|\}$  and  $\{|\widehat{f}|\}$  when we apply Young's inequality to the first term in the right-hand side of (77), by taking the  $\ell^1$ -norm of  $\{|\widehat{\langle \partial_x \rangle^\tau v}|\}$  and the  $\ell^2$ -norm of  $\{|\widehat{f}|\}$ ; then the  $\ell^1$ -norm of  $\{|\widehat{\langle \partial_x \rangle^\tau v}|\}$  is estimated again by the first inequality in (79)

$$\|\{|\widehat{\langle \partial_x \rangle^\tau v}|\}\|_{\ell^1} \leq C_\sigma \|\langle \partial_x \rangle^\tau v\|_{H^\sigma(\mathbb{T})} \leq C_\sigma \|v\|_{H^{\tau+\sigma}(\mathbb{T})}.$$

To obtain the last inequality (71), the  $\ell^2$ -norm of the sequence  $\{|\widehat{\langle \partial_x \rangle^\tau v} * \widehat{f}|\}$  in the right-hand side of (77) is estimated by Young's inequality as

$$\|\{|\widehat{\langle \partial_x \rangle^\tau v} * \widehat{f}|\}\|_{\ell^2} \leq \|\{|\widehat{\langle \partial_x \rangle^\tau v}|\}\|_{\ell^{4/3}} \|\{|\widehat{f}|\}\|_{\ell^{4/3}}; \quad (80)$$

then recalling that for every  $p \in ]1, 2]$  a positive constant  $C_p$  exists such that

$$\|\{|\widehat{f}|\}\|_{\ell^p} \leq C_p \|f\|_{H^{1/2}(\mathbb{T})}, \quad \forall f \in H^{1/2}(\mathbb{T}), \quad (81)$$

the  $\ell^{4/3}$ -norms in the right-hand side of (80) are estimated as

$$\|\{|\widehat{\langle \partial_x \rangle^\tau v}|\}\|_{\ell^{4/3}} \leq C \|\langle \partial_x \rangle^\tau v\|_{H^{1/2}(\mathbb{T})} \leq C \|v\|_{H^{\tau+1/2}(\mathbb{T})}, \quad \|\{|\widehat{f}|\}\|_{\ell^{4/3}} \leq C \|f\|_{H^{1/2}(\mathbb{T})} \quad (82)$$

(that is (81) with  $p = \frac{4}{3}$ ). Then (71) follows from gathering the estimates (80), (82) and repeating for the rest the same calculations as above.  $\square$

## REFERENCES

- [1] H. Beirão da Veiga. Data dependence in the mathematical theory of compressible inviscid fluids. *Arch. Rational Mech. Anal.*, 119(2):109–127, 1992.
- [2] H. Beirão da Veiga. Perturbation theorems for linear hyperbolic mixed problems and applications to the compressible Euler equations. *Comm. Pure Appl. Math.*, 46(2):221–259, 1993.
- [3] H. Beirão da Veiga. Singular limits in compressible fluid dynamics. *Arch. Rational Mech. Anal.*, 128(4):313–327, 1994.
- [4] J.K. Hunter and J.B. Thoo. On the weakly nonlinear Kelvin-Helmholtz instability of tangential discontinuities in MHD. *J. Hyperbolic Differ. Equ.*, 8(4):691–726, 2011.
- [5] T. Kato. Quasi-linear equations of evolution, with applications to partial differential equations. pages 25–70. Lecture Notes in Math., Vol. 448. Springer, Berlin, 1975.
- [6] T. Kato. Nonlinear equations of evolution in Banach spaces. In *Nonlinear functional analysis and its applications, Part 2*, volume 45 of *Proc. Sympos. Pure Math.*, pages 9–23. Amer. Math. Soc., 1986.
- [7] A. Morando, P. Secchi and P. Trebeschi. Existence of approximate current-vortex sheets near the onset of instability. arXiv:1601.03337.
- [8] A. Morando, P. Secchi and P. Trebeschi. Approximate current-vortex sheets near the onset of instability. *J. Math. Pures Appl.*, 2015, to appear. arXiv:1511.00811.
- [9] P. Secchi. On the singular incompressible limit of inviscid compressible fluids. *J. Math. Fluid Mech.*, 2(2):107–125, 2000.

DICATAM, SEZIONE DI MATEMATICA,

UNIVERSITÀ DI BRESCIA,

VIA VALOTTI, 9, 25133 BRESCIA, ITALY

E-mail address: `alessandro.morando@unibs.it`, `paolo.secchi@unibs.it`, `paola.trebeschi@unibs.it`